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Representations of Lie algebras

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Abstract
Most significant contributions to the Representation Theory of Lie algebras by the members of the research group of IME-USP and their collaborators are described. The focus is made on the Gelfand-Tsetlin theories, representations of affine Kac-Moody algebras, related vertex algebras and Lie algebras of vector fields.

Keywords Galois order · Gelfand-Tsetlin module · Affine Lie algebra · Witt algebra · Vector field · Krichever-Novikov algebra

1 Introduction

Representation theory of Lie algebras is an active mainstream branch of Mathematics which plays an increasingly important role in different areas of modern science. In particular, representations of Lie algebras are of fundamental use in geometry, mathematical physics, topology, combinatorics, number theory, knot theory etc. This theory was a focus of the research group "Non-associative algebras, their representations, identities and relations" of IME-USP for more than 20 years. The ultimate goal is to develop a general framework and new methodologies to address challenging classification and structure problems using algebraic, geometric and combinatorics techniques. We will describe the most notable results obtained by the members of the research group on the representations of Lie algebras. Another paper of this volume will focus onstructure results of non-associative algebras.

First we consider Gelfand-Tsetlin theory for finite-dimensional Lie algebras and related structures which have attracted a considerable interest in the last 40 years

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after the pioneering work of I.Gelfand and M.Tsetlin [63]. It can be viewed in a more general context of Harish-Chandra categories [28], which play an important role in the representation theory. The classification of simple objects in these categories is of prime importance.

As an attempt to unify the representation theories of the universal enveloping algebra of the full linear Lie algebra $\mathfrak{gl}_n$ and of the generalized Weyl algebras, a concept of Galois orders was introduced in [52]. The idea grows out of the classical concept of a Gelfand-Tsetlin basis for finite-dimensional representations of simple Lie algebras. This provided a new framework for the study of representation of various classes of algebras. Next we will focus on representation theory of infinite-dimensional Lie algebras. We start with affine Kac-Moody algebras, which is one of the most active and fascinating branches in the theory of Lie algebras. These algebras and related structures of vertex algebras, Yangians etc. serve as a motivation for mathematical models in quantum field theory and give rise to fundamentally new phenomena. Two closely related mathematical models for the states of elementary particles or strings are called Verma modules and Fock spaces (also called *free field realizations*). This motivated the theory of vertex algebras and construction of different free field realizations of the affine Kac-Moody algebra and related structures. There is a growing interest in the study of non-highest weight representations of affine Lie algebras with an expectation of their importance for non-rational affine vertex algebras. In particular, *relaxed Verma modules* have connections to the representation theory of conformal vertex algebras and conformal field theories, e.g. $N = 2$ conformal field theory [32] and $N = 4$ conformal field theory [1]. The study of positive energy representations of simple affine vertex algebras can be reduced to the representations of corresponding finite-dimensional Lie algebras using the Zhu's functor which allows to construct new families of simple modules for vertex algebras [94].

One of the main reasons that the theory of affine Kac-Moody algebras has become such a popular area of research is that they (untwisted ones) have a realization given by the one dimensional central extension of the *loop algebra*, $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algera. Replacing $\mathbb{C}[t, t^{-1}]$ by any commutative associative algebra R we obtain a new infinite-dimensional Lie algebra. In particular, a class of Krichever-Novikov Lie algebras corresponds to the case when R is an algebra of meromorphic functions on a Riemann surface with finite number of poles. The case of Laurent polynomials corresponds to the Riemann sphere $\mathbb{C} \cup \infty$ with poles allowed only in $\{0, \infty\}$. For the theory of Krichever-Novikov Lie algebras we refer to [86]. If instead of the sphere with two punctures we consider any complex algebraic curve of genus 0 with a fixed set of n distinct points where the poles are allowed, then we obtain *n-point* affine Lie algebras. We obtain a class of *elliptic* Lie algebras if genus is 1 and *superelliptic* Lie algebras if genus is greater than 1.

Integrable systems arising from the Landau-Lifshitz differential equation [27] are described by the action on the space of solutions of certain infinite-dimensional Lie algebra, called *the DJKM algebra*, which is an example of a n-point Lie algebra. The universal central extension of the DJKM algebra can be described in terms of certain families of polynomials. It led to the discovery of a new family of non-classical orthogonal polynomials satisfying the fourth order differential equation [26]. Other families of polynomials that appear in the description of the central extension are examples of the associated Jacobi polynomials of Ismail-Wimp.

Finally, we will discuss recent advances in the representation theory of Lie algebras of vector fields on affine varieties, whose history goes back to Sophus Lie. They are defined as derivation algebras of the ring of polynomial functions on an irreducible affine variety over an algebraically closed field of characteristic 0. These algebras is a source of examples of simple infinite-dimensional Lie algebras which are connected with the symmetries of geometric structures and with the symmetries of systems with infinitely many degrees of freedom.

The following members of the research group made a contribution to the theory and related topics: A.Bianchi (USP and Unesp) A.Bueno (USP and UFMG), T.Bunke (USP), M.Cardoso (USP), C.Duque (USP), V.Futorny (USP), D.Grantcharov (University of Texas), C.Gomes (USP and UFRN), A.Grishkov (USP), M.Guerrini (USP), J.Hartwig (USP and Iowa State University), K.Iusenko (USP), I.Kashuba (USP), L.Krizka (USP), R.Martins (USP and UNESP), G.Monsalve (USP and UFAM), O.Morales (USP), W.Mutis (USP and University of Nariño), A.Oliveira (USP), E.Ramirez (USP and UFABC), H.Rocha (USP), F. dos Santos (USP), A.Sargeant (USP and UFOP), J.Schwarz (USP), E.Vishnyakova (USP and UFMG), B.Wilson (USP), E.Wilson (USP), A.Zaidan (USP) P.Zadunaisky (USP and University of Buenos Aires), J.Zhang (USP and Central China Normal University).

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2 Gelfand-Tsetlin theories

The field $\Bbbk$ is assumed to be algebraically closed of characteristic 0. All rings are assumed to be $\Bbbk$-algebras.

2.1 Harish-Chandra modules

Let U be an associative algebra over the field $\Bbbk$, $\Gamma \subset U$ a noetherian commutative subalgebra, $\operatorname{Specm}\Gamma$ the set of maximal ideals of Γ. Following [28] we say that M is a *Harish-Chandra module* (with respect to Γ) if M is a finitely generated U-module such that

$$M = \bigoplus_{\mathbf{m}\in \operatorname{Specm}\Gamma} M(\mathbf{m}),$$

where

$$M(\mathbf{m}) \;=\; \{x \in M \mid \exists k,\ \mathbf{m}^k x = 0\}.$$

The *support* of a Harish-Chandra module M is a subset of $\operatorname{Specm}\Gamma$ consisting of those $\mathbf{m}$ for which $M(\mathbf{m}) \neq 0$.

We denote by $\mathbb{H}(U,\Gamma)$ the category of all Harish-Chandra U-modules. *Gelfand-Tsetlin theories* study the categories $\mathbb{H}(U,\Gamma)$ with maximal commutative Γ.

We will assume that Γ is a *Harish-Chandra* subalgebra of U [28]: for any $u \in U$ theΓ-bimodule $\Gamma u \Gamma$ is finitely generated both as a left and as a right Γ-module. This notion proved to be very powerful in the study of Harish-Chandra modules. Note that $M \in \mathbb{H}(U,\Gamma)$ if and only if every finitely generated Γ-submodule of M has finite length. For other equivalent conditions see [54].

For a subset $D \subset \operatorname{Specm}\Gamma$ denote by $\mathbb{H}(U,\Gamma,D)$ the full subcategory in $\mathbb{H}(U,\Gamma)$ consisting of modules whose support is a subset of D. For an element $u \in U$ set

$$X_u \;=\; \{(\mathbf{m},\mathbf{n}) \in \operatorname{Specm}\Gamma \times \operatorname{Specm}\Gamma \mid \Gamma/\mathbf{n} \;\text{ is a subquotient of }\; (\Gamma u \Gamma)/(\Gamma u \mathbf{m})\}.$$

Denote by Δ the minimal equivalence on $\operatorname{Specm}\Gamma$ containing sets X_u for all $u \in U$ and set $\Delta(U,\Gamma)$ to be the set of Δ–equivalence classes on $\operatorname{Specm}\Gamma$.

We have

$$\mathbb{H}(U,\Gamma) = \bigoplus_{D\in\Delta(U,\Gamma)} \mathbb{H}(U,\Gamma,D).$$

Let $\Gamma_{\mathbf{m}} = \lim_{\leftarrow k} \Gamma/\mathbf{m}^k$ be the completion of Γ with respect to a maximal ideal $\mathbf{m}$. The following category $\mathcal{A}_{U,\Gamma}$ was introduced in [28], see also [54] for more details. The set of objects $\operatorname{Ob}\mathcal{A}_{U,\Gamma}$ consists of maximal ideals of Γ while the space of morphisms $\mathcal{A}_{U,\Gamma}(\mathbf{m},\mathbf{n})$ is defined as

$$\begin{aligned}\mathcal{A}_{U,\Gamma}(\mathbf{m},\mathbf{n}) \;&=\; \lim_{\leftarrow n,m} U/(\mathbf{n}^n U + U\mathbf{m}^m) = \\ &= \lim_{\leftarrow n,m} \Gamma/\mathbf{n}^n \otimes_\Gamma U \otimes_\Gamma \Gamma/\mathbf{m}^m.\end{aligned} \tag{1}$$

The category $\mathcal{A}_{U,\Gamma}$ splits into the sum of full subcategories,

$$\mathcal{A}_{U,\Gamma} = \bigoplus_{D\in\Delta(U,\Gamma)} \mathcal{A}_{U,\Gamma}(D),$$

where $\mathcal{A}_{U,\Gamma}(D)$ denotes the restriction of $\mathcal{A}_{U,\Gamma}$ on D.

The category $\mathcal{A}_{U,\Gamma}$ is endowed with the topology of the inverse limit and the category of $\Bbbk$-vector spaces ($\Bbbk -$ mod) with the discrete topology. Consider the category $\mathcal{A}_{U,\Gamma} - \text{mod}_d$ of *discrete* modules (that is, continuous functors) $M : \mathcal{A}_{U,\Gamma} \longrightarrow \Bbbk - \text{mod}$.

Proposition 2.1 ([[28], Theorem 17]) *Categories* $\mathcal{A}_{U,\Gamma} - \text{mod}_d$ *and* $\mathbb{H}(U,\Gamma)$ *are equivalent.*

To study simple modules in the category $\mathbb{H}(U,\Gamma)$ one may try to parametrize them by simple Γ-modules, or equivalently by maximal ideals of Γ. The restriction functor from the category $\mathbb{H}(U,\Gamma)$ to the category of torsion Γ-modules induces a map Φ from $\text{Specm}\,\Gamma$ to the set of isomorphism classes $Irr(U)$ of simple U-modules in $\mathbb{H}(U,\Gamma)$. Given a maximal ideal $\mathbf{m} \in \text{Specm}\,\Gamma$, $\Phi(\mathbf{m})$ consist of those simple $V \in \mathbb{H}(U,\Gamma)$ such that $V(\mathbf{m}) \neq 0$ (or left maximal ideals of U which contain $\mathbf{m}$).

In the case when both U and Γ are commutative and $\Gamma \subset U$ is an integral extension then any prime ideal of Γ lifts to a prime ideal of U. Moreover, if U is finitely generated module over Γ then number of liftings is finite for every prime ideal of Γ. This is reflected in the Hilbert-Noether theorem when U is the symmetric algebra of a finite-dimensional vector space V and Γ is the subalgebra of G-invariants of U for a finite subgroup G of $GL(V)$.

The non-emptiness of the fibers of Φ is related to the freeness of U over Γ as a right module. A remarkable result of Kostant [74] shows that the universal enveloping algebra U of a reductive Lie algebra is free over the center Γ of U. Any central character of Γ defines a block of $\mathbb{H}(U,\Gamma)$ of modules with such central character, and each block contains infinitely many simple objects.

Ovsienko [82] introduced a technique to study the freeness of universal enveloping algebras of Lie algebras over polynomial subalgebras. A graded version of this technique was developed in [51] which allows to apply this technique to a large class of *special filtered* associative algebras. An associative algebra U endowed with an increasing filtration is special if any element can be written uniquely as a linear combination of ordered monomials in some fixed generators of U and if the associated graded algebra is polynomial. We have

Theorem 2.1 ([51]) *Let U be a special filtered $\Bbbk$-algebra. Let $g_1, \cdots, g_t \in U$ be mutually commuting elements whose graded images form a complete intersection for the associated graded algebra of U. Then U is a free left (right) $\Bbbk[g_1, \dots, g_t]$-module.*

This result was used to show the freeness of the restricted Yangian of $\mathfrak{gl}_n$ over its center [51] and the freeness of the restricted Yangian of $\mathfrak{gl}_2$ over its Gelfand-Tsetlin subalgebra. The problem of the freeness is related to the equidimensionality of certain *Gelfand-Tsetlin varieties*. In the case of restricted Yangian of $\mathfrak{gl}_n$ this variety was studies in [5], see also [6].

2.2 Galois algebras

Theory of Galois rings (orders) was developed in [52] to deal with the problem of the finiteness of the fibers $\Phi(\mathbf{m})$ of maximal ideals of commutative subalgebras.

Let R be a ring, $\mathcal{M}$ a monoid acting on R by ring automorphisms and $R * \mathcal{M}$ is the skew monoid ring.

Let Γ be an integral domain, K the field of fractions of Γ and L a finite Galois extension of K with the Galois group $G = \mathrm{Gal}(L/K)$. Consider the action of G by conjugation on Aut (L). Let $\mathcal{M}$ be any G-invariant submonoid of Aut (L). We assume the following property of $\mathcal{M}$: if $m_1, m_2 \in \mathcal{M}$ and $m_1|_K = m_2|_K$ then $m_1 = m_2$. Denote by $\mathcal{K} = (L * \mathcal{M})^G$ the subring of invariants.

Definition 2.1 A finitely generated Γ-subring U of $\mathcal{K}$ is called a *Galois ring over* Γ if $UK = KU = \mathcal{K}$.

We assume that all Galois rings are $\mathbb{k}$-algebras. In this case we say that a Galois ring is a Galois algebra over Γ.

Example 2.1 Let $U = \Gamma(\sigma, a)$ be a *generalized Weyl algebra* of rank 1 ([3]), where Γ is a unital integral domain, $a \in \Gamma$, σ an automorphism of Γ of infinite order. It is generated over Γ by X and Y such that $X\gamma = \sigma(\gamma)X$, $Y\gamma = \sigma^{-1}(\gamma)Y$, $YX = a$, $XY = \sigma(a)$. Let K be the field of fractions of Γ and $\mathcal{M} \simeq \mathbb{Z}$ is a subgroup of Aut Γ generated by σ. Then U can be embedded into the skew group algebra $K * \mathbb{Z}$ when $X \mapsto \sigma$ and $Y \mapsto a\sigma^{-1}$. Clearly, U is a Galois algebra over Γ. Note that $U \simeq \Gamma * \mathbb{Z}$ if a is invertible in Γ.

A Galois ring U over Γ is *right (respectively left) Galois order* [52], if for any finite-dimensional right (respectively left) K-subspace $W \subset U[S^{-1}]$ (respectively $W \subset [S^{-1}]U$), $W \cap U$ is a finitely generated right (respectively left) Γ-module. A Galois ring is *Galois order* if it is both right and left Galois order.

This is a natural non-commutative generalization of the classical concept of order in skew group rings. If Γ is finitely generated and U is a Galois order over Γ then Γ is a Harish-Chandra subalgebra of U. Moreover, if $\mathcal{M}$ is a group then Γ is a maximal commutative subalgebra of U.

Examples of Galois orders include the generalized Weyl algebras over integral domains with infinite order automorphisms (e.g. the Weyl algebras, quantized Weyl algebras, the quantum plane, the q-deformed Heisenberg algebra, the Witten-Woronowicz algebra) [59], the universal enveloping algebra of $\mathfrak{gl}_n$ over the Gelfand-Tsetlin subalgebra [28], finite W-algebras [49] among the others. We also have

Theorem 2.2 ([52], Theorem 5.2, (2)) *If a Galois ring U over a noetherian domain Γ is projective as a right* (*left*) Γ-*module then U is a right* (*left*) *Galois order.*

Further examples of Galois orders were recently constructed in [65]. Set $\mathcal{K} = (L * \mathcal{M})^G$ and

$$\mathcal{K}_\Gamma = \{x \in \mathcal{K} \mid x(\gamma) \in \Gamma \text{ for all } \gamma \in \Gamma\}.$$

Then $\mathcal{K}_\Gamma$ is a Galois order over Γ in $\mathcal{K}$ [65, Theorem 2.21]. Principal Galois orders are Galois subrings of $\mathcal{K}_\Gamma$. This includes orthogonal Gelfand-Tsetlin algebras [65, Theorem 4.6], quantum orthogonal Gelfand-Tsetlin algebras [65, Theorem 5.6], Coulomb branches [91] and *rational Galois orders*. Rational Galois orders are attached to an arbitrary finite reflection group and a set of difference operators with rational function coefficients. The parabolic subalgebras of finite W-algebras of type A are examples of rational Galois orders [65, Theorem 1.2].

New examples of Galois orders can be obtained by considering the invariants of generalized Weyl algebras under the linear actions of finite groups [58–60]. Let G_m be the cyclic group of order m. Fix a primitive mth root of unity w. Then G_m acts on the Weyl algebra A_1 as follows: $\partial \to w\partial; x \to w^{-1}x$, where x and ∂ are the standard generators of A_1, $[\partial, x] = 1$. Denote by $A_1^{G_m}$ the subalgebra of invariants under this action. Similarly, one defines the invariant subalgebra $A_n^{G_m^{\otimes n}}$ of the nth Weyl algebra. For $m \geq 1, n \geq 1, p|m$ let $A(m, p, n)$ be the subgroup of $G_m^{\otimes n}$ of all elements $(h_1, \ldots, h_n)$ such that $(h_1 h_2 \ldots h_n)^{m/p} = id$. The groups $G(m,p,n) = A(m,p,n) \rtimes S_n$, where the symmetric group S_n permutes the entries in $A(m, p, n)$, are non-exceptional complex reflection groups. Let $\mathcal{A}_n$ be the alternating subgroup of S_n. We have

Theorem 2.3 ([59], Theorem 1) *Let* $W \in \{G_m^{\otimes n}, G(m,1,n), \mathcal{A}_n, m \geq 1, n \geq 1\}$. *Then* A_n^W *is a principal Galois order over* $\Gamma = \Bbbk[t_1, \ldots, t_n]^W$, *where* $t_i = \partial_i x_i, i = i, \ldots, n$.

Analogs of Theorem 2.3 also hold for invariants of differential operators on the torus [59, Theorem 3].

It was shown in [53] that Galois orders have a nice theory of Harish-Chandra modules. Let $U \subset (L * \mathcal{M})^G$ be a Galois ring over Γ and $\mathbf{m}$ a maximal ideal of Γ. Let $\bar{\mathbf{m}}$ be any lifting of $\mathbf{m}$ to the integral closure of Γ in L. The cardinality $|\mathbf{m}|$ of the stabilizer of $\bar{\mathbf{m}}$ in $\mathcal{M}$ depends only on $\mathbf{m}$.

Theorem 2.4 ([53], Theorem A, , Theorem 8) *Let* Γ *be a finitely generated commutative domain,* $U \subset (L * \mathcal{M})^G$ *a right Galois order over* Γ, $\mathbf{m} \in \operatorname{Specm}\Gamma$ *with finite* $|\mathbf{m}|$.

(i) *The fiber* $\Phi(\mathbf{m})$ *is non-empty. Moreover, If* U *is a Galois order over* Γ, *then the fiber* $\Phi(m)$ *is finite.*
(ii) *Let* U *be a Galois order over* Γ, *where* Γ *is a normal noetherian* $\Bbbk$*-algebra, and* $M \in \mathbb{H}(U, \Gamma)$ *is simple* U*-module. Then all subspaces* $M(\mathbf{n})$ *and the number of isomorphism classes of simple modules* N, *such that* $N(\mathbf{m}) \neq 0$, *are bounded.*

2.3 Gelfand-Tsetlin $\mathfrak{gl}_n$-modules

In this subsection we assume $\Bbbk = \mathbb{C}$ and consider Gelfand-Tsetlin representations for the Lie algebra $\mathfrak{gl}_n$ consisting of all $n \times n$ complex matrices with the standard basis of elementary matrices $e_{i,j}$, $1 \le i,j \le n$. For each $k \leqslant n$ denote by $\mathfrak{gl}_k$ the Lie subalgebra of $\mathfrak{gl}_n$ spanned by $\{e_{ij} \mid i,j = 1,\dots,k\}$. We have the following embeddings of Lie subalgebras

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_n.$$

We have corresponding embeddings $U_1 \subset U_2 \subset \cdots \subset U_n$ of the universal enveloping algebras $U_k = U(\mathfrak{gl}_k)$, $1 \le k \le n$. Set $U = U_n$.

Let Z_k be the center of U_k. This is the polynomial algebra generated by the following elements:

$$c_{ks} = \sum_{(i_1,\dots,i_s)\in\{1,\dots,k\}^s} e_{i_1 i_2} e_{i_2 i_3} \dots e_{i_s i_1}, \tag{2}$$

$s = 1,\dots,k$.

Let Γ be the *Gelfand-Tsetlin subalgebra* of $U(\mathfrak{gl}_n)$ generated by the centers Z_k, $k = 1,\dots,n$. The generators c_{ks}, $k = 1,\dots,n$, $s = 1,\dots,k$ are algebraically independent [93].

Let Λ be the polynomial algebra in the variables $\{\lambda_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\}$. Consider the embedding $\pi : \Gamma \longrightarrow \Lambda$ such that

$$c_{ks} \mapsto \sum_{i=1}^{k} (\lambda_{ki} + k - 1)^s \prod_{j\neq i} \left(1 - \frac{1}{\lambda_{ki} - \lambda_{kj}}\right).$$

One can easily check that $\pi(c_{ks})$ is a symmetric polynomial in Λ of degree s in variables $\lambda_{k1},\dots,\lambda_{kk}$. Let $G = \prod_{i=1}^{n} S_i$ be the product of symmetric groups. Then G acts naturally on Λ where S_k permutes the variables $\lambda_{k1},\dots,\lambda_{kk}$, $k = 1,\dots,n$. The image of Γ, $\pi(\Gamma)$, coincides with the subalgebra of G-invariant polynomials in Λ which we identify with Γ.

Consider the Harish-Chandra category $H(U,\Gamma)$. The modules of $H(U,\Gamma)$ are called *Gelfand-Tsetlin modules*. If $M \in H(U,\Gamma)$ then

$$M = \bigoplus_{\mathsf{m}\in \operatorname{Specm}\Gamma} M(\mathsf{m}),$$

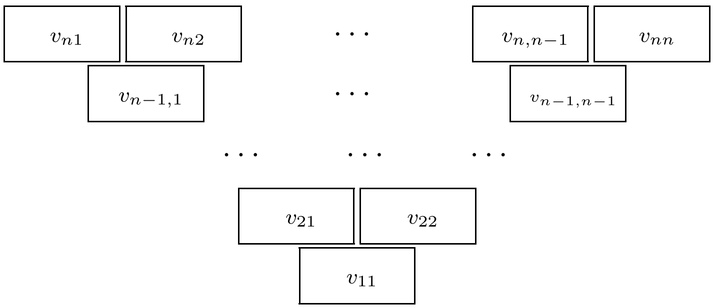
where $M(\mathsf{m}) = \{v \in M \mid \mathsf{m}^k v = 0 \text{ for some } k \ge 0\}$.

For a Gelfand-Tsetlin module $M(\mathsf{m}) \in H(U,\Gamma)$ and $\mathbf{m} \in \operatorname{Specm}\Gamma$ we call the dimension of $M(\mathsf{m})$ *the Gelfand-Tsetlin multiplicity* of $\mathbf{m}$.

A classical result of Gelfand and Tsetlin [63] provides an explicit basis for all simple finite-dimensional $\mathfrak{gl}_n$-modules. This basis is given by special Gelfand-Tsetlin tableaux. Identify $\mathbb{C}^{\frac{n(n+1)}{2}}$ with $T_n(\mathbb{C}) = \mathbb{C}^n \times \mathbb{C}^{n-1} \times \dots \times \mathbb{C}$ and write every vector $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$ in the following form:

$$v = (v_{n1}, ..., v_{nn} | v_{n-1,1}, ..., v_{n-1,n-1} | \cdots | v_{21}, v_{22} | v_{11}).$$

Then we define the following *Gelfand-Tsetlin tableau* $T(v)$:



For $v = (v_{ij})_{j \leq i=1}^{n} \in T_n(\mathbb{C})$ consider the complex vector space $V(T(v))$ spanned by the set

$$v + T_{n-1}(\mathbb{Z}) = \{ v + w \mid w = (w_{ij})_{j \leq i=1}^{n}, w_{ij} \in \mathbb{Z}, w_{nk} = 0\,, k = 1, \dots, n \}.$$

A Gelfand-Tsetlin tableau $T(v)$ is *standard* if $v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq k \leq n-1$.

If $L(\lambda)$ is the simple finite-dimensional $\mathfrak{gl}_n$-module of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$, then the set of all standard tableaux $T(v)$ with fixed top row $v_{ni} = \lambda_i - i + 1$, $i = 1, \dots, n$ gives a *Gelfand-Tsetlin basis* of $L(\lambda)$. Moreover, one can explicitly write the action of of the generators of $\mathfrak{gl}(n)$ on these basis tableaux [63].

To every tableau $T(v)$ we associate the maximal ideal $\mathbf{m}_v$ of Γ generated by $c_{ij} - \gamma_{ij}(v)$, where

$$\gamma_{mk}(v) := \sum_{i=1}^{m} (v_{mi} + m - 1)^k \prod_{j \neq i} \left(1 - \frac{1}{v_{mi} - v_{mj}} \right).$$

Let $\mathcal{M} = \mathbb{Z}^{\frac{n(n-1)}{2}}$ be the free abelian group generated by δ^{ij}, $1 \leqslant j \leqslant i \leqslant n-1$, where $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{k\ell}$ are zero, $1 \leq j \leq i \leq n-1$ and let $G = S_n \times S_{n-1} \times \cdots \times S_1$. Identify $\mathcal{M}$ with $T_{n-1}(\mathbb{Z})$ and consider its action on $T_n(\mathbb{C})$ by translations: $\delta^{ij} \cdot v = \delta^{ij} + v$. Also, consider the action of G on $T_n(\mathbb{C})$, where S_k acts on the kth row:

$$\sigma(v_{k1}, \dots, v_{kk}) := (v_{k,\sigma^{-1}(1)}, \dots, v_{k,\sigma^{-1}(k)}).$$

Let L be the field of fractions of Λ and K the field of fractions of Γ. Then $K = L^G$ and G is the Galois group of the field extension $K \subset L$. The following map $\tau : U \to (L * \mathcal{M})^G$ is a homomorphism of algebras, where

$$\tau(e_{mm}) = e_{mm} * e,\ \tau(e_{m\,m+1}) = \sum_{i=1}^{m} a^{+}_{mi}\delta^{mi},\ \tau(e_{m+1\,m}) = \sum_{i=1}^{m} a^{-}_{mi}(\delta^{mi})^{-1},$$

$$a^{\pm}_{mi} = \mp\frac{\prod_j(\lambda_{m\pm1,j} - \lambda_{mi})}{\prod_{j\neq i}(\lambda_{mj} - \lambda_{mi})},$$

and e is the unit of $\mathcal{M}$.

This embedding defines on U the structure of a Galois order over Γ [52, Proposition 7.2].

Homomorphism τ can also be used to construct infinite-dimensional Gelfand-Tsetlin modules which have a basis parametrized by Gelfand-Tsetlin tableaux and with the action of Γ determined by the entries of tableaux as in (2.3). Such Gelfand-Tsetlin modules are called *tableau modules*.

If the action of the generators of $\mathfrak{gl}_n$ in a tableau Gelfand-Tsetlin module is given by the classical Gelfand-Tsetlin formulas as in finite-dimensional modules then such module is called *standard tableau module*. Families of standard tableau modules were studied in [28, 37, 56, 62, 77, 78].

In particular, if v is *generic*, that is $v_{rs} - v_{rt} \notin \mathbb{Z}$ for any $r < n$ and all possible $s \neq t$, then $V(T(v))$ is *generic* standard tableau module [37]. All simple generic Gelfand-Tsetlin modules were described in [37].

If v contains a pair (v_{ki_j}, v_{ki_s}) such that $k > 1$ and $v_{ki_j} - v_{ki_s} \in \mathbb{Z}$, then v (and $T(v)$) is *singular*. Finite-dimensional $\mathfrak{gl}_n$-modules are examples of tableau Gelfand-Tsetlin modules with singular tableaux. Families of infinite-dimensional tableau Gelfand-Tsetlin modules with singular tableaux were considered in [38, 40, 56, 62, 77, 78]. In particular, the problem of constructing singular standard tableau Gelfand-Tsetlin modules was solved in [56, Theorem II] for any tableau $T(w)$ satisfying special *FRZ-condition*. This includes all known examples of standard tableau modules. For any such tableau $T(w)$ there exists a unique simple standard tableau Gelfand-Tsetlin $\mathfrak{gl}_n$ -module V_w such that $V_w(\mathbf{m}_w) \neq 0$ and all Gelfand-Tsetlin multiplicities of maximal ideals of Γ in the support of V_w equal 1. A combinatorial approach developed in [56] allows to explicitly construct a large class of simple tableau modules with singular tableaux. Moreover, it was shown for $n \leq 4$ (and conjectured for all n) that modules constructed in [56] exhaust all simple standard tableau Gelfand-Tsetlin modules.

A systematic study of singular modules was initiated in [37]. We say that v is *singular of index* $m \geq 2$ if:

(i) there exists a row $k, 1 < k < n$, and m entries $v_{ki_1}, \dots, v_{ki_m}$ on this row such that $v_{ki_j} - v_{ki_s} \in \mathbb{Z}$ for all $j, s \in \{1, \dots, m\}$;
(ii) m is maximal with the property (i).

A tableau Gelfand-Tsetlin module structure on $V(T(v))$ for singular v of index $m = 1$ was established in [[37], Theorem 4.11]. In this case the module $V(T(v))$ is not standard tableau module, its basis contains *derivative* tableaux and the Gelfand-Tsetlin multiplicities are bounded by 2 (see also [88] and [92]). The structure of $V(T(v))$ was described in [64]. The case of arbitrary singularity of index $m = 2$ was studied in [39], where any number of singular pairs (but not singular triples) and multiple singular pairs in the same row were allowed. The case of an arbitrary singularity was solved in [89] (p-singularity) and [83] (arbitrary singularity). Finally, an alternative geometric approach developed in [90] led to the classification of simple Gelfand-Tsetlin modules. It showed a deep connection between the Gelfand-Tsetlin theory and Coulomb branches.

Remark 2.1

- Singular tableau Gelfand-Tsetlin modules have beautiful connections with Schubert calculus and Postnikov polynomials [41] and with tensor product categorification and KLRW algebras [75];
- It is still a conjecture that any simple Gelfand-Tsetlin module V with $V(\mathbf{m}_v) \neq 0$ is isomorphic to a subquotient of $V(T(v))$ for any singular v. It is known to be true for $n = 2$ and $n = 3$, and in the 1-singular case. In particular, there is a complete explicit classification of all simple Gelfand-Tsetlin $\mathfrak{gl}_3$-modules [36].

Denote GT the category of all Gelfand-Tsetlin $\mathfrak{gl}_3$-modules, and for each orbit ζ in $\mathbb{C}^{\frac{n(n+1)}{2}}$ of the action of $T_{n-1}(\mathbb{Z})\#G$ denote by GT_ζ the full subcategory of GT consisting of modules with support in ζ. Then

$$\mathsf{GT} = \bigoplus_{\zeta \in \mathbb{C}^{\frac{n(n+1)}{2}}/(T_{n-1}(\mathbb{Z})\#G)} \mathsf{GT}_\zeta.$$

Given $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$ define the *graph* $\Omega(v)$ whose vertices are pairs of indices $\{(k,i) \mid 1 \leq i \leq k \leq n\}$, and there is an edge between (k, i) and (l, j) if and only if $v_{k,i} - v_{l,j} \in \mathbb{Z}$ and $|k-l| \leq 1$. We will say that $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$ is in *normal form* if whenever $v_{k,a} - v_{k,b} \in \mathbb{Z}$ for some $a \leq b \leq k \leq n$, then $v_{k,i} - v_{k,j} \in \mathbb{Z}_{\geq 0}$ for all $a \leq i < j \leq b$. We will say that $\bar{v} \in \mathbb{C}^{\frac{n(n+1)}{2}}$ is a *seed* if v is in a normal form and for (k, i) and (l, j) from the same connected component of $\Omega(v)$ the following holds: if $k, l < n$ then $\bar{v}_{k,i} = \bar{v}_{l,j}$, while if $l = n$ then $\bar{v}_{k,i} \leq \bar{v}_{n,j}$. To explain these concepts we consider an example below of an element $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$, a normal form of v, a seed and the corresponding graph. It is assumed that the set $\{1, a, b, c, \dots\} \subset \mathbb{C}$ is linearly independent over $\mathbb{Z}$.

$$\begin{array}{ccccccccc} 1 & & a+1 & & a & & b & & 0 \\ & a & & b-1 & & b & & a+1 & \\ & & c & & c+1 & & c & & \\ & & & a & & a-1 & & & \\ & & & & a+1 & & & & \end{array}$$

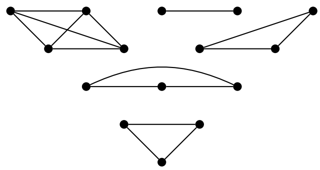
An element $v \in \mathbb{C}^{\frac{n(n+1)}{2}}$

$$\begin{array}{ccccccccc} 1 & & 0 & & a+1 & & a & & b \\ & a+1 & & a & & b & & b-1 & \\ & & c+1 & & c & & c & & \\ & & & a & & a-1 & & & \\ & & & & a+1 & & & & \end{array}$$

A normal form in $G \cdot v$

$$\begin{array}{ccccccccc} a+1 & & a & & 1 & & 0 & & b \\ & a & & a & & b-1 & & b-1 & \\ & & c & & c & & c & & \\ & & & a-1 & & a-1 & & & \\ & & & & a-1 & & & & \end{array}$$

A seed in $(T_{n-1}(\mathbb{Z})\#G) \cdot v$



The graph

Fix a seed $\overline{v}$ and set $\zeta = \zeta_{\overline{v}} \in \mathbb{C}^{\frac{n(n+1)}{2}}/(T_{n-1}(\mathbb{Z})\#G)$. Let $G_{\overline{v}} < G$ be the stabilizer of $\overline{v}$. For $z \in T_{n-1}(\mathbb{Z})$ such that $\overline{v}+z$ is in normal form set $(G_{\overline{v}})_z < G_{\overline{v}}$ to be the stabilizer of z. If $M \in GT_\zeta$ then

$$M = \oplus_{z \in T_{n-1}(\mathbb{Z})} M(\mathbf{m}_{\overline{v}+z})$$

The following *FO inequality* was established in [53, Theorem 4.12(c)]. It gives an upper bound for the Gelfand-Tsetlin multiplicities of any simple Gelfand-Tsetlin module:

$$\dim M(\mathbf{m}_{\overline{v}+z}) \leq \frac{|G_{\overline{v}}|}{|(G_{\overline{v}})_z|}. \tag{3}$$

It was conjectured in [53, Remark 5.4] that this inequality is sharp. This was shown to be true in [41, Theorems 8.3, 8.5] and in [29, Theorems 10,11].

We call the *essential support* of M the set of all z for which the equality holds in (3). In fact, (3) gives a sharp bound in each subcategory GT_ζ:

Theorem 2.5 (Strong Futorny-Ovsienko Conjecture) [42] *Let* $\overline{v}$ *be a seed,* $\zeta = \zeta_{\overline{v}}$. *Then*

- (i) *The module* $V(T(\overline{v}))$ *has a simple socle* V_{soc};
- (ii) *The essential support of* V_{soc} *is nonempty. It consists of integral points of a finite union of polyhedral rational cones, at least one of which is of maximal possible rank* $\frac{n(n-1)}{2}$;
- (iii) *The maximal Gelfand-Tsetlin multiplicity of a character in* GT_ζ *is* $|G_{\overline{v}}|$, *and this is attained at the socle* V_{soc};

(iv) *For any $\overline{v}+z$ in the essential support of V_{soc}, the module V_{soc} is the unique simple Gelfand-Tsetlin module having $\overline{v}+z$ in its support.*

2.4 Representations of *W*-algebras

Finite W-algebras are certain family of algebras that are associated with nilpotent orbits in semisimple Lie algebras. These algebras are closely connected with Yangian theory and with affine W-algebras and attract a growing interest in their representation theory. Let $\pi=(p_1,\dots,p_m)$ be a sequence of integers such that $1\le p_1\le\dots\le p_m$ and $p_1+\dots+p_m=n$. Then π defines a finite W-algebra $U=W(\pi)$ of type $\mathbf{A}$. Set $\pi_k=(p_1,\dots,p_k)$, $k=1,\dots,m$ and consider the corresponding finite W-algebras $W(\pi_k)$. Then

$$W(\pi_1)\subset\dots\subset W(\pi_m)=W(\pi).$$

Let Γ be a subalgebra of $W(\pi)$ generated by the centers of all $W(\pi_k)$, $k=1,\dots,m$. The center Z of $W(\pi)$ is polynomial algebra in $e=p_1+\dots+p_m$ variables, while Γ is a polynomial algebra in $d=mp_1+(m-1)p_2+\dots+2p_{m-1}+p_m$ variables which is usually called the *Gelfand-Tsetlin subalgebra* of $W(\pi)$. If $m=e$ and $p_1=\dots=p_m=1$ then $W(\pi)$ is isomorphic to the universal enveloping algebra $U(\mathfrak{gl}_n)$.

Theorem 2.6 [49, *Theorem* 3.6] *$W(\pi)$ is a Galois order over Γ.*

Theorem 2.6 implies that $W(\pi)$ has a nice theory of Harish-Chandra modules in $\mathbb{H}(W(\pi),\Gamma)$ (see also [48]). Moreover, it allows to prove the Gelfand-Kirillov conjecture for $W(\pi)$ [49, Theorem I]. An important ingredient of the proof is the Noncommutative Noether's problem on the invariants in the skew fields of the Weyl algebras with respect to linear group actions. We have the following remarkable fact

Theorem 2.7 [60] *For any field $\Bbbk$ of zero characteristic and any linear action of a finite group G, if the quotient variety $\mathbb{A}^n(\Bbbk)/G$ is rational then the Noncommutative Noether's problem holds.*

A large family of new simple modules for an arbitrary finite W-algebra of type A was explicitly constructed in [57]. A basis of these *relation* modules is given by the Gelfand-Tsetlin tableaux whose entries satisfy certain sets of relations. Also, the simplicity of tensor product of any number of highest weight modules with generic highest weight was established.

2.5 Generalized Gelfand-Tsetlin theories

Let Γ be a commutative noetherian Harish-Chandra subalgebra of an associative algebra U and assume U to be finitely generated over Γ. Then Γ has the maximal torsion (that is all generators of Γ have torsion) on Harish-Chandra modules in $\mathbb{H}(U,\Gamma)$.

Thus $\mathbb{H}(U,\Gamma)$ serves as a starting point in the stratification of the whole module category $U-$ Mod by prime ideals of Γ [54, 55]. The subcategories of this stratification are the *generalized Harish-Chandra categories* consisting of U-modules with smaller torsion.

Recall that a *torsion theory* over Γ is a pair $(\mathcal{T},\mathcal{F})$ of full subcategories of $\Gamma-$ mod such that:

(1) $\mathcal{T}$ consists of all Γ-modules T such that $\mathrm{Hom}_{\Gamma}(T,F)=0$, for all $F\in\mathcal{F}$,
(2) $\mathcal{F}$ consists of all Γ-modules F for which $\mathrm{Hom}_{\Gamma}(T,F)=0$, for all $T\in\mathcal{T}$

Define a transfinite ascending chain of subsets $(Z_i)_{i\ \mathrm{ordinal}}$ by setting $Z_0=\operatorname{Specm}\Gamma$ and $Z_i=\bigcup_{j<i}Z_j$, in case i is a limit ordinal, and $Z_i=Z_{i-1}\cup\operatorname{Max}(\operatorname{Spec}\Gamma\setminus Z_{i-1})$ in case i is nonlimit. Here $\operatorname{Max}A$ denotes the set of maximal elements of A. Then for each $\mathbf{p}\in\operatorname{Spec}\Gamma$ there is a minimal ordinal $i_{\mathbf{p}}$ such that $\mathbf{p}\in Z_{i_{\mathbf{p}}}$. The nonlimit ordinal $i_{\mathbf{p}}$ is the *coheight* $cht(\mathbf{p})$ of $\mathbf{p}$.

We have a transfinite ascending chain of torsion classes $\mathcal{T}_0\subseteq\mathcal{T}_1\subseteq\ldots\subseteq\mathcal{T}_i\subseteq\ldots$ such that $\Gamma-\mathrm{Mod}=\bigcup_{i\leq\delta}\mathcal{T}_i$ for some ordinal δ. If M is a Γ-module then for uniquely determined ordinal i we have $M\in\mathcal{T}_i$ and $M\notin\mathcal{T}_j$, for all $j<i$.

For a torsion class $\mathcal{T}$ in $\Gamma-$ Mod and for a U-module M denote the torsion Γ-submodule of M in $\mathcal{T}$ by $T(M)$. If $T(M)$ is a U-submodule of M for every U-module M then we say that a torsion theory $(\mathcal{T},\mathcal{F})$ in $\Gamma-$ Mod is *liftable* to $U-$ Mod.

The following result shows that the module category $U-$ Mod has a stratification with respect to the coheight of prime ideals.

Theorem 2.8 [55, *Theorem A*] *Let $i\geq 0$ be an integer. The torsion theory associated to the subset $Z_i\subset\operatorname{Spec}\Gamma$ of prime ideals of coheight $\leq i$ is liftable to U. For any simple U-module M all associated prime ideals of M in* $\operatorname{Spec}\Gamma$ *have the same coheight.*

3 Representations of infinite-dimensional Lie algebras

3.1 Affine Lie algebras

In 1967 V.Kac and R.Moody extended the generators and relations construction of finite-dimensional simple Lie algebras to a new important class of infinite-dimensional Lie algebras, now appropriately called *Kac-Moody algebras*, by relaxing the condition on the Cartan matrix to be positive definite [68]. Representation theory of these algebras is a rich source of interesting research with numerous applications. Specially important is the family of *affine* Kac-Moody algebras which correspond to the case of positive semidefinite generalized Cartan matrix. Their representations are relevant to theory of theta functions, modular forms, vertex algebras, the Boson-Fermion correspondence and soliton equations, to name just a few. Description of simple modules for affine Kac-Moody algebras is a very challenging problem of prime importance.

Let $\mathfrak{G}$ be an affine Kac-Moody algebra with a 1-dimensional center $Z = \mathbb{C}c$ and a fixed Cartan subalgebra $\mathfrak{H}$.

Classification of simple $\mathfrak{G}$-modules is known in various subcategories of weight modules, i.e. those on which the subalgebra $\mathfrak{H}$ acts diagonally [19, 33–35, 66, 4, 61] but remains open in general. A key ingredient in the construction of simple module for affine Lie algebras is a parabolic induction. The long standing conjecture [34], Conjecture 8.1 states that parabolic induction reduces the classification of simple modules to the classification of so-called *dense* modules (with maximal possible support). This conjecture was proved for $A_1^{(1)}$ [33], for $A_2^{(2)}$, [18] and for all affine Lie algebras in the case of modules with finite-dimensional weight spaces and non-zero action of c [61].

Parabolic subalgebras of affine Lie algebras are of two types: those with a finite-dimensional Levi subalgebra and those with an infinite-dimensional one. Let $\mathfrak{P} \subset \mathfrak{G}$ be a parabolic subalgebra such that $\mathfrak{P} = \mathfrak{l} \oplus \mathfrak{n}$ is a Levi decomposition with an infinite-dimensional Levi factor $\mathfrak{l}$. Then $\mathfrak{l}$ contains the Heisenberg subalgebra G of $\mathfrak{G}$ generated by all imaginary root subspaces of $\mathfrak{G}$. Let $\mathfrak{l}^0$ be the Lie subalgebra of $\mathfrak{l}$ generated by all real root subspaces and $\mathfrak{H}$, $G(\mathfrak{l})$ a subalgebra of $\mathfrak{l}^0$ generated by its imaginary root subspaces. Then $\mathfrak{l} = \mathfrak{l}^0 + G_{\mathfrak{l}}$, where $G_{\mathfrak{l}} \subset G$ is the orthogonal complement of $G(\mathfrak{l})$ in G with respect to the Killing form, that is $G = G(\mathfrak{l}) + G_{\mathfrak{l}}$, $[G_{\mathfrak{l}}, \mathfrak{l}^0] = 0$ and $\mathfrak{l}^0 \cap G_{\mathfrak{l}} = \mathbb{C}c$. For any positive integer k, denote $\mathfrak{G}_k$ the Lie subalgebra of $\mathfrak{G}$ generated by the root subspaces $\mathfrak{G}_{\pm k\delta}$. We say that a $\mathfrak{G}_k$ -module V is $\mathfrak{G}_{k\delta}$-surjective (respectively $\mathfrak{G}_{-k\delta}$-surjective) if for any two elements $v_1, v_2 \in V$ there exist $v \in V$ and $u_1, u_2 \in U(\mathfrak{G}_{k\delta})$ (respectively, $u_1, u_2 \in U(\mathfrak{G}_{-k\delta})$) such that $v_i = u_i v$, $i = 1, 2$. A $G_{\mathfrak{l}}$-module T is *admissible* if for any positive integer k, any its cyclic $\mathfrak{G}_k$-submodule $T' \subset T$ is $\mathfrak{G}_{k\delta}$-surjective or $\mathfrak{G}_{-k\delta}$-surjective.

A simple weight $\mathfrak{l}$-module is called *tensor* if it is isomorphic to a tensor product of a simple weight $\mathfrak{l}^0$-module S with a $\mathbb{Z}$-graded simple $G_{\mathfrak{l}}$-module T with the same scalar action of c. A tensor module $S \otimes T$ is called admissible if T is admissible $G_{\mathfrak{l}}$-module.

If N is a weight $\mathfrak{l}$-module then consider the induced module

$$ind_N(\mathfrak{P}, \mathfrak{G}) = U(\mathfrak{G}) \otimes_{U(\mathfrak{P})} N,$$

where $\mathfrak{n}N = 0$. The following result allows to construct explicitly a large family of simple $\mathfrak{G}$-modules from simple $\mathfrak{l}$-modules. Particular cases of this theorem were proved in [4, 43].

Theorem 3.1 [44, *Theorem* 1] *Let* $\mathfrak{P} = \mathfrak{l} \oplus \mathfrak{n} \subset \mathfrak{G}$ *be a parabolic subalgebra of* $\mathfrak{G}$ *with infinite-dimensional Levi factor* $\mathfrak{l}$. *Then* $ind_N(\mathfrak{P}, \mathfrak{G})$ *is a simple* $\mathfrak{G}$*-module for any simple admissible tensor* $\mathfrak{l}$*-module* N *with a non-zero scalar action of* c.

Fock space realizations of affine Kac–Moody algebras were introduced in [90] for affine sl_2 and were generalized in [31] for all untwisted affine Lie algebra.

These remarkable boson realizations, called *Wakimoto modules*, plays an important role in the conformal field theory for the Wess-Zumino-Witten models.

Different free field realizations of affine Lie algebras were constructed in [7, 20, 24, 70, 79, 80], yielding in particular explicit constructions of *imaginary Verma modules* and *intermediate Wakimoto modules* for affine sl_n. A uniform construction for an arbitrary untwisted affine Kac–Moody algebra which includes all cases above, was given in [47, Theorem 3.3]. It was motivated by geometric representation theory for generalized flag manifolds of finite-dimensional semisimple Lie groups.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, κ an invariant symmetric bilinear form on $\mathfrak{g}$, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ a parabolic subalgebra of $\mathfrak{g}$ with the Levi factor $\mathfrak{l}$, $\bar{\mathfrak{u}}$ is the opposite radical of $\mathfrak{u}$. Consider the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_\kappa = \mathfrak{g}(\!(t)\!) \oplus \mathbb{C}c$ with the commutation relations

$$[a \otimes f(t), b \otimes g(t)] =$$

$$= [a, b] \otimes f(t)g(t) - \kappa(a, b)Res_{t=0}(f(t)dg(t))c.$$

Consider the natural parabolic subalgebra $\mathfrak{p}_{nat}$ of $\widehat{\mathfrak{g}}$

$$\mathfrak{p}_{nat} = \mathfrak{l}_{nat} \oplus \mathfrak{u}_{nat},$$

where

$$\mathfrak{l}_{nat} = \mathfrak{l} \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!) \oplus \mathbb{C}c$$

and the nilradical $\mathfrak{u}_{nat}$ of $\mathfrak{p}_{nat}$ and the opposite nilradical $\bar{\mathfrak{u}}_{nat}$ are given by

$$\mathfrak{u}_{nat} = \mathfrak{u} \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!) \qquad \text{and} \qquad \bar{\mathfrak{u}}_{nat} = \bar{\mathfrak{u}} \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!).$$

We have the triangular decomposition of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{g}} = \bar{\mathfrak{u}}_{nat} \oplus \mathfrak{l}_{nat} \oplus \mathfrak{u}_{nat}.$$

Let $\sigma : \mathfrak{p}_{nat} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{p}_{nat}$ such that $\sigma(c) = id_V$. Then the *generalized imaginary Verma module* of level k is the induced module

$$\mathbb{M}_{\sigma,\kappa,\mathfrak{p}}(V) = \mathrm{Ind}_{\mathfrak{p}_{nat}}^{\widehat{\mathfrak{g}}} V = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{p}_{nat})} V.$$

Consider the commutative $\mathbb{C}$-algebra $\mathcal{K} = \mathbb{C}(\!(t)\!)$, $\Omega_{\mathcal{K}} = \mathbb{C}(\!(t)\!)\, dt$. For a finite-dimensional complex vector space V we define $\mathcal{K}(V) = V \otimes_{\mathbb{C}} \mathcal{K}$ and $\Omega_{\mathcal{K}}(V^*) = V^* \otimes_{\mathbb{C}} \Omega_{\mathcal{K}}$. A natural pairing between $\Omega_{\mathcal{K}}(V^*)$ and $\mathcal{K}(V)$ allows us to identify the restricted dual space of $\mathcal{K}(V)$ with $\Omega_{\mathcal{K}}(V^*)$. Set $\mathrm{Pol}\Omega_{\mathcal{K}}(V^*)$ for the polynomials on $\Omega_{\mathcal{K}}(V^*)$.

The following theorem establishes an isomorphism between the geometric realization of the affine Lie algebra $\widehat{\mathfrak{g}}$ and the corresponding generalized imaginary Verma $\widehat{\mathfrak{g}}$-module

Theorem 3.2 [47, *Theorem* 3.15] *Let* (σ, V) *be a continuous* $\mathfrak{p}_{nat}$*-module. Then we have an isomorphism of* $\widehat{\mathfrak{g}}$*-modules*:

$$\mathbb{M}_{\sigma,\kappa,\mathfrak{p}}(V) \simeq \mathrm{Pol}\Omega_{\mathcal{K}}(\bar{\mathfrak{u}}^*) \otimes_{\mathbb{C}} V.$$

For a $\mathfrak{g}$-module E set

$$\mathbb{M}_{\kappa,\mathfrak{g}}(E) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}[[t]]\oplus\mathbb{C}c)} E,$$

where $\mathfrak{g} \otimes t\mathbb{C}[[t]]E = 0$ and c acts on E as the identity. Then we get a functor $\mathbb{M}_{\kappa,\mathfrak{g}}$ from the category of $\mathfrak{g}$-modules to the category of *relaxed highest weight* $\widehat{\mathfrak{g}}$-modules. The module $\mathbb{M}_{\kappa,\mathfrak{g}}(\mathbb{C})$ has a natural structure of a vertex algebra, called the *universal affine vertex algebra* associated with $\mathfrak{g}$, which we denote $V^\kappa(\mathfrak{g})$. The *simple affine vertex algebra* $V_\kappa(\mathfrak{g})$ is the unique simple graded quotient of $\mathbb{M}_{\kappa,\mathfrak{g}}(\mathbb{C})$. There is a one-to-one correspondence between simple positive energy representations of $V_\kappa(\mathfrak{g})$ and simple admissible modules over the Zhu's algebra $A(V_\kappa(\mathfrak{g}))$ of $V_\kappa(\mathfrak{g})$ [94], where

$$A(V_\kappa(\mathfrak{g})) \cong U(\mathfrak{g})/I_\kappa$$

for some two-sided ideal I_κ of $U(\mathfrak{g})$. We also have $A(V^\kappa(\mathfrak{g})) \cong U(\mathfrak{g})$. This correspondence allows to construct new families of simple representations of these vertex algebras. In particular, new families of simple modules were constructed for the universal affine vertex algebra of $\mathfrak{sl}_n$ in [2]. This approach has also been exploited in [50, 72, 73] and [45]. In particular, the localization technique and the Wakimoto functors were used in [45] to construct *relaxed Wakimoto modules* for affine vertex algebras. The twisting functor T_α on the category of $\widehat{\mathfrak{g}}$-modules is assigned to a positive root α of $\mathfrak{g}$ and is defined as follows

$$T_\alpha(M) = (D_\alpha U(\widehat{\mathfrak{g}}_\kappa)/U(\widehat{\mathfrak{g}})) \otimes_{U(\widehat{\mathfrak{g}})} M,$$

for a $\widehat{\mathfrak{g}}$-module M, where $D_\alpha U(\widehat{\mathfrak{g}})$ is the localization of $U(\widehat{\mathfrak{g}})$ relative to the multiplcative set $\{f_\alpha^k \mid k \in \mathbb{Z}_{\geq 0}\} \subset U(\widehat{\mathfrak{g}})$.

There exists a natural isomorphism

$$T_\alpha \circ \mathbb{M}_{\kappa,\mathfrak{g}} \to \mathbb{M}_{\kappa,\mathfrak{g}} \circ T_\alpha^{\mathfrak{g}}$$

of functors, where $T_\alpha^{\mathfrak{g}}$ is the twisting functor for $\mathfrak{g}$ assigned to α. In particular, for a Verma $\mathfrak{g}$-module $M(\lambda)$ of $\mathfrak{g}$ of highest weight λ we have

$$T_\alpha(\mathbb{M}_{\kappa,\mathfrak{g}}(M(\lambda))) \simeq \mathbb{M}_{\kappa,\mathfrak{g}}(W(\lambda,\alpha)),$$

where $W(\lambda,\alpha) \in \mathbb{H}(U(\mathfrak{g}),\Gamma_\alpha)$, where Γ_α is commutative subalgebra generated by the Cartan subalgebra of $\mathfrak{g}$ and by the Casimir element of root α. It was shown in [46] that $W(\lambda,\alpha)$ has finite Γ_α-multiplicities Moreover, it has finite Γ-multiplicities for any commutative subalgebra Γ of $U(\mathfrak{g})$ containing Γ_α.

The Feigin-Frenkel homomorphism between the universal affine vertex algebra and the tensor product of the Weyl vertex algebra with the Heisenberg vertex algebra gives an explicit free field construction of Wakimoto modules. It was used in [45] to obtain a free field realization of relaxed Verma modules.

Let $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$, $B = \{f_\alpha; \alpha \in \Delta_+\}$ a root basis of $\bar{\mathfrak{n}}$, $\{x_\alpha; \alpha \in \Delta_+\}$ linear coordinate functions on $\bar{\mathfrak{n}}$ with respect to B, where Δ_+ is the set of positive roots for $\mathfrak{g}$. Consider the Weyl algebra $\mathcal{A}_{\bar{\mathfrak{n}}}$ of $\bar{\mathfrak{n}}$ generated by $\{x_\alpha, \partial_{x_\alpha}; \alpha \in \Delta_+\}$ and the Weyl algebra $\mathcal{A}_{\mathcal{K}(\bar{\mathfrak{n}})}$ is topologically generated by the set $\{x_{\alpha,n}, \partial_{x_{\alpha,n}}; \alpha \in \Delta_+, n \in \mathbb{Z}\}$ with the canonical commutation relations. The algebra $\mathcal{A}_{\mathcal{K}(\bar{\mathfrak{n}})}$ has a natural $\mathbb{Z}$-grading with $\mathcal{A}_{\mathcal{K}(\bar{\mathfrak{n}}),0} \simeq \mathcal{A}_{\bar{\mathfrak{n}}}$.

For an $\mathcal{A}_{\bar{\mathfrak{n}}}$-module N define the induced module

$$\mathbb{M}_{\mathcal{K}(\bar{\mathfrak{n}})}(N) = \mathcal{A}_{\mathcal{K}(\bar{\mathfrak{n}})} \otimes_{\mathcal{A}_{\mathcal{K}(\bar{\mathfrak{n}}),0} \otimes_{\mathbb{C}} \mathcal{A}_{\mathcal{K}(\bar{\mathfrak{n}}),+}} N,$$

where $\mathcal{A}_{\mathcal{K}(\bar{\mathfrak{n}}),+}$ acts trivially on N. Moreover, if E is an $\mathfrak{h}$-module then define *relaxed Wakimoto* $\widehat{\mathfrak{g}}_\kappa$-module

$$\mathbb{W}_{\kappa,\mathfrak{g}}(N \otimes_{\mathbb{C}} E) = \mathbb{M}_{\mathcal{K}(\bar{\mathfrak{n}})}(N) \otimes_{\mathbb{C}} \mathbb{M}_{\kappa-\kappa_c,\mathfrak{h}}(E),$$

where κ_c is the critical invariant symmetric bilinear form on $\mathfrak{g}$ (cf. [45, Lemma 2.4]). This defines the *Wakimoto functor* from the category of modules over $\mathcal{A}_{\bar{\mathfrak{n}}} \otimes U(\mathfrak{h})$ to the category of smooth $\widehat{\mathfrak{g}}_\kappa$-modules. We have the following properties of the Wakimoto functor.

Theorem 3.3 [45, *Theorem B*] *Let* $\alpha \in \Delta_+$, $\lambda \in \mathfrak{h}^*$.

- There exists a natural isomorphism

$$T_\alpha \circ \mathbb{W}_{\kappa,\mathfrak{g}} \simeq \mathbb{W}_{\kappa,\mathfrak{g}} \circ T_\alpha^{\mathfrak{g}}$$

  of functors, where $T_\alpha^{\mathfrak{g}}$ is the twisting functor for $\mathfrak{g}$ assigned to α. In particular,

$$T_\alpha(\mathbb{W}_{\kappa,\mathfrak{g}}(M(\lambda))) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(W(\lambda, \alpha));$$

- If the Verma $\widehat{\mathfrak{g}}_\kappa$-module $\mathbb{M}_{\kappa,\mathfrak{g}}(M(\lambda))$ is simple, then

$$\mathbb{M}_{\kappa,\mathfrak{g}}(W(\lambda, \alpha)) \simeq \mathbb{W}_{\kappa,\mathfrak{g}}(W(\lambda, \alpha)).$$

  Hence, we have a free field realization of simple relaxed Verma module $\mathbb{M}_{\kappa,\mathfrak{g}}(W(\lambda, \alpha))$;
- If $\langle \lambda + \rho, \beta^\vee \rangle \notin -\mathbb{N}$ for all $\beta \in \Delta_+$, then

$$\mathbb{M}_{\kappa_c,\mathfrak{g}}(W(\lambda, \alpha)) \simeq \mathbb{W}_{\kappa_c,\mathfrak{g}}(W(\lambda, \alpha)).$$

Write $\kappa = k\kappa_0$ for $k \in \mathbb{C}$, where κ_0 is the normalized $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ satisfying

$$\kappa_{\mathfrak{g}} = 2h^\vee \kappa_0,$$

where $\kappa_{\mathfrak{g}}$ is the Killing form and $h^\vee$ is the dual Coxeter number.

The number k is *admissible* if the affine vertex algebra $V_\kappa(\mathfrak{g})$ is admissible as a highest weight module over $\hat{\mathfrak{g}}$ in the sense of [69]. For example, if $\mathfrak{g} = \mathfrak{sl}_{n+1}$ then k is admissible if and only if

$$k + n = \frac{p}{q} - 1 \quad \text{with } p, q \in \mathbb{N},\ (p, q) = 1,\ p \geq n + 1.$$

A $\mathfrak{g}$-module M is called admissible of level k if k is an admissible number and M is an $A(V_\kappa(\mathfrak{g}))$-module. Explicit construction in terms of Gelfand–Tsetlin tableaux of all admissible simple highest weight $\widehat{\mathfrak{sl}}_{n+1}$-modules and all admissible simple $\mathfrak{sl}_{n+1}$ -modules induced from $\mathfrak{sl}_2$ in the minimal nilpotent orbit was obtained in [50].

3.2 Krichever-Novikov algebras and orthogonal polynomials

Let R is a commutative $\mathbb{C}$-algebra and $\mathfrak{g}$ be a simple complex Lie algebra. The universal central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g} \otimes R$ is the Lie algebra $(\mathfrak{g} \otimes R) \oplus \Omega^1_R/dR$, where Ω^1_R/dR is the space of Kähler differentials modulo exact forms and

$$[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)\overline{fdg}, \quad [x \otimes f, \omega] = 0$$

for $x, y \in \mathfrak{g}$, $f, g \in R$, and $\omega \in \Omega^1_R/dR$ [71]. Unlike in the affine case, the universal central extension of $\mathfrak{g} \otimes R$ need not to be one dimensional for a general R. When R is the algebra of meromorphic functions on a Riemann surface and with a fixed finite number of points where the poles are allowed then $\mathfrak{g} \otimes R$ is a *Krichever-Novikov algebra* [76]. In the genus 0 case we obtain the *n-point algebras* with $R = \mathbb{C}[(t - x_1)^{-1}, \dots, (t - x_N)^{-1}]$. The universal central extension of such algebras was described by in [14]. In particular, the universal central extension of a 4-point Lie algebra can be given explicitly in terms of Gegenbauer orthogonal polynomials [15], while in the *elliptic* case with $R = \mathbb{C}[x, x^{-1}, y \mid y^2 = 4x^3 - g_2x - g_3]$, the universal central extension is described in terms of Pollaczek polynomials [16].

The ring $R = \mathbb{C}[t, t^{-1}, u]|u^2 = t^4 - 2ct^2 + 1]$, $c \in \mathbb{C} \setminus \{\pm 1\}$ corresponds to the DJKM algebra which was introduced in [27] in the study of the solutions of the Landau-Lifshitz equation which describes time evolution of magnetism in solids. The universal central extension of the DJKM algebra was described explicitly in [23] in terms of certain polynomials $P_k(c)$ in c which satisfy the recursion relation

$$(6 + 2k)P_k(c) = 4kcP_{k-2}(c) - 2(k - 3)P_{k-4}(c)$$

for $k \geq 0$. Depending of the initial conditions we obtain four families of polynomials, two of which are Gegenbauer polynomials and the other two are given by elliptic integrals. Assume for example that $P_{-3}(c) = P_{-2}(c) = P_{-1}(c) = 0$ and $P_{-4}(c) = 1$. Then the generating function is defined via an elliptic integral:

$$P_{-4}(c, z) := \sum_{k \geq 0} P_{-4,k-4}(c) z^k = z\sqrt{1 - 2cz^2 + z^4} \int \frac{4cz^2 - 1}{z^2(z^4 - 2cz^2 + 1)^{3/2}}\, dz.$$

The second elliptic case corresponds to the following initial conditions: $P_{-1}(c) = P_{-3}(c) = P_{-4}(c) = 0$ and $P_{-2}(c) = 1$. It turned out that both families of polynomials satisfy the fourth order differential equation:

Theorem 3.4 [26]

- The polynomials $P_n = P_{-4,n}$ satisfy the following differential equation:

$$16(c^2-1)^2P_n^{(iv)} + 160c(c^2-1)P_n''' - 8(c^2(n^2-4n-46) - n^2 + 4n + 22)P_n'' \\ -24c(n^2-4n-6)P_n' + (n-4)^2n^2P_n = 0.$$

- The polynomials $Q_n = P_{-2,n}$ satisfy the following differential equation

$$16(c^2-1)^2Q_n^{(iv)} + 160c(c^2-1)Q_n''' - 8(c^2(n^2-4n-42) - n^2 + 4n + 18)Q_n'' \\ -24c(n^2-4n-2)Q_n' + (n-6)(n-2)^2(n+2)Q_n = 0.$$

Polynomials $P_{-4,n}$ is a special case of associated ultraspherical polynomials, which implies their orthogonality with respect to a certain weight function. On the other hand polynomials $P_{-2,n}$ are not associated ultraspherical polynomials. Nevertheless, their orthogonality with some weight function can be shown by using the Favard's theorem (see [26] for details). Hence we obtain

Theorem 3.5 *Polynomials $P_{-4,n}$ and $P_{-2,n}$ are non-classical orthogonal polynomials.*

In particular, polynomials $P_{-2,n}$ is a new family of orthogonal polynomials. It is natural to expect that the universal central extensions of other Krichever-Novikov algebras might lead to more families of non-classical orthogonal polynomials. In the superelliptic case when $R = \mathbb{C}[t, t^{-1}, u | u^m = p(t)],\ p(t) \in \mathbb{C}[t]$ the central extension was described recently in [85].

Generalizing Wakimoto's construction the free field type realizations of the elliptic Lie algebra and of the DJKM algebra were constructed in [17] and [25] respectively for $\mathfrak{g} = sl_2$. Free field realization 3-point and 4-point algebras were constructed in [22] and [21] respectively.

4 Vector fields on algebraic varieties

Lie algebras of vector fields on affine varieties are objects of fundamental importance. Nevertheless, their general theory, in particular their representation theory, at large is still undeveloped.

Let $X \subset \mathbb{A}^n$ be an irreducible affine algebraic variety over an algebraically closed field $\Bbbk$ of characteristic zero, and let $I_X = \langle g_1, \dots, g_m \rangle$ be the ideal of all functions that vanish on X. Let $A_X := \Bbbk[x_1, \dots, x_n]/I_X$ be the algebra of polynomial functions on X. Denote by $\mathcal{V}_X := \mathrm{Der}_{\Bbbk}(A_X)$ the Lie algebra of polynomial vector fields on X, which is the Lie algebra of derivations of $A = A_X$. The Lie algebra $\mathcal{V}_X$ is simple if and only if X is a smooth variety [67, 87] (see also [10]). A classical example is the first Witt algebra which is the Lie algebra of polynomial vector fields on a circle. Its universal central extension is the famous Virasoro algebra which plays a crucial role in quantum field theory. Its generalization is the Lie algebra of vector fields on a torus

$$W_n = \mathrm{Der}(\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]) = \bigoplus_{p=1}^{n} Ad_p,$$

where $A = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $d_1 = t_1 \frac{\partial}{\partial t_1}, \ldots, d_n = t_n \frac{\partial}{\partial t_n}$ span a Cartan subalgebra H of W_n. Setting $t^r = t_1^{r_1} \ldots t_n^{r_n}$ for $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n$, the Lie bracket in W_n is defined as follows:

$$[t^r d_i, t^s d_j] = s_i t^{r+s} d_j - r_j t^{r+s} d_i, \quad i,j = 1, \ldots, n; r, s \in \mathbb{Z}^n.$$

Simple Harish-Chandra modules for the first Witt algebra W_1 were classified in [81]. Numerous attempt were made to extend this classification to Lie algebras of polynomial vector fields on n-dimensional torus. This was finally achieved in [8] where all simple modules in $\mathbb{H}(U(W_n), U(H))$ with finite multiplicities were classified using the new concept of AW_n-modules which are W_n-modules and at the same time A-modules with some compatibility condition:

$$x(fv) = (xf)v + f(xv), \quad f \in A, x \in W_n, v \in M.$$

The theory of AW_n-modules is a generalization of a D-module theory. We recall the definition of important class of tensor modules. Fix a finite-dimensional $\mathfrak{gl}_N$-module U and $\gamma \in \mathbb{C}^n$. Define the module of *tensor fields*

$$T(U, \gamma) = q^\gamma \mathbb{C}[q_1^{\pm 1}, \ldots, q_n^{\pm 1}] \otimes U$$

with the action

$$t^r d_i(q^\mu \otimes u) = \mu_i q^{\mu+r} \otimes u + \sum_{k=1}^{n} r_k q^{\mu+r} \otimes E_{ki} u,$$

where $r \in \mathbb{Z}^n, \mu \in \gamma + \mathbb{Z}^n, i = 1, \ldots, n$.

In particular, the modules of differential forms are tensor fields modules: $q^\gamma \Omega^k(\mathbb{T}^n) = T(\Lambda^k \mathbb{C}^n, \gamma)$. They form the de Rham complex

$$q^\gamma \Omega^0(\mathbb{T}^n) \xrightarrow{d} q^\gamma \Omega^1(\mathbb{T}^n) \xrightarrow{d} \ \ldots \ \xrightarrow{d} q^\gamma \Omega^n(\mathbb{T}^n).$$

If module of tensor fields is not isomorphic to one of the members of this de Rham complex then it is simple [30].

Another class of W_n-modules consists of highest weight type modules. Choose a $\mathbb{Z}$-grading on W_n by degree in t_n. Then a zero component W_n^0 is a semidirect product of W_{n-1} with an abelian ideal. We take a W_{n-1}-module of tensor fields for W_{n-1} and define the action of the abelian ideal by multiplication rescaled with a complex parameter β, yielding a W_n^0-module $T(U,\gamma,\beta)$. Setting $W_n^+T(U,\gamma,\beta)=0$ we construct the induced W_n-module $M(U,\gamma,\beta)$ which has a unique simple quotient $L(U,\gamma,\beta)$ with finite weight multiplicities. For $g\in\mathfrak{gl}$ define a twisting $L(U,\gamma,\beta)^g$, which corresponds to a change in the $\mathbb{Z}$-grading on W_n. Vertex operator realizations of simple W_n-modules of the highest weight type were given in [9].

The classification of simples Harish-Chandra W_n-modules is as follows:

Theorem 4.1 [8, *Theorem* 1.1] *Every simple module in* $\mathbb{H}(U(W_n),U(H))$ *with finite multiplicities is isomorphic to either* $T(U,\gamma)$, *where* U *is different from* $\Lambda^k\mathbb{C}^n$, $k=0,\dots,n$, *or* $L(U,\gamma,\beta)^g$, *or a submodule* $d\Omega^k(\beta)\subset\Omega^{k+1}(\beta)$ *for* $0\le k<n$ *and* $\beta\in\mathbb{C}^n$.

Representations of $\mathcal{V}_X$ for $X=\mathbb{A}^n_{\mathbf{k}}$ were studied by Rudakov [84] and for $X=\mathbb{S}^2$ in [12]. A systematic study of representations of the Lie algebras $\mathcal{V}_X$ for arbitrary smooth affine varieties X was initiated in [11]. Developing ideas of [12] and [8] two families of simple $\mathcal{V}_X$-modules were constructed: *Rudakov modules* and *gauge modules*.

Let $p\in X$, U a finite-dimensional simple $\mathfrak{gl}_N$-module, where N is the dimension of X. Then U is a module over the smash product $A\#U(\mathfrak{gl}_N)$, where $f\cdot u:=f(p)u$ for $f\in\mathbb{A}$ and $u\in U$. The *Rudakov module* is an induced module

$$R_p(U):=A\#U(\mathcal{V}_X)\otimes_{A\#U(\mathfrak{gl}_N)}U,$$

this generalizes highest weight type modules for W_n.

Gauge modules are generalizations of tensor modules determined by the gauge fields (see [11] for details) $B_i:A_{(h)}\otimes U\to A_{(h)}\otimes U$, $i=1,\dots,N$, where $A_{(h)}$ is the localization of A by a minor h of the Jacobian matrix of I_X of maximal rank, and U is a finite-dimensional $\mathfrak{gl}_N$-module. Then $\mathbb{A}_{(h)}\otimes U$ is a $Der(\mathbb{A}_{(h)})$-module with the action

$$\left(f\frac{\partial}{\partial t_i}\right)\cdot(g\otimes u)=f\frac{\partial g}{\partial t_i}\otimes u+fg\otimes B_iu+\sum_{k\in\mathbb{Z}_+^N\setminus\{0\}}\frac{1}{k!}g\frac{\partial^k f}{\partial t^k}\otimes\rho\left(t^k\frac{\partial}{\partial t_i}\right)u,$$

where $f,g\in A_{(h)}$ and $u\in U$.

An $A\mathcal{V}_X$-module M is a gauge module if it is isomorphic to an $A\mathcal{V}_X$-submodule of $A_{(h)}\otimes U$ of finite rank over A for any minor h of maximal rank.

Theorem 4.2 *If* X *is smooth then* $R_p(U)$ *and gauge modules are simple* $\mathbb{A}\mathcal{V}_X$*-module* [11]. *Moreover, If* U *is not an exterior power of the dual to natural module* (*resp. natural module*), *then* $R_p(U)$ (*resp. a gauge module*) *is simple as a* $\mathcal{V}_X$*-module* [13].

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